Slope and Elasticity in Regression Models

Consider the following 5 models:

i. $\ln Y_t = \beta_1 + \beta_2 \ln X_t + u_t$

ii. $Y_t = \beta_1 + \beta_2 X_t + u_t$

iii. $\ln Y_t = \beta_1 + \beta_2 X_t + u_t$

iv. $Y_t = \beta_1 + \beta_2 \ln X_t + u_t$

v. $Y_t = \beta_1 + \beta_2\ 1/X_t + u_t$

Remember the Implicit Function Theorem, if $G(X_1, X_2, \ldots, X_N, Y) = 0$

$$\frac{dY}{dX_k} = -\frac{\frac{dG}{dX_k}}{\frac{dG}{dY}}$$

i. $\boldsymbol{\ln Y_t = \beta_1 + \beta_2 \ln X_t + u_t}$

$$\ln Y_t - \beta_1 - \beta_2 \ln X_t - u_t = 0$$

$$\frac{dY_t}{dX_t} = -\frac{\frac{dG}{dX_t}}{\frac{dG}{dY_t}} = -\frac{-\beta_2 * \frac{1}{X_t}}{\frac{1}{Y_t}} = \frac{\beta_2\, Y_t}{X_t}$$

$$\frac{dY_t}{dX_t} = \frac{\beta_2\, Y_t}{X_t}$$

$$\beta_2 = \frac{dY_t}{dX_t}\frac{X_t}{Y_t} = \frac{\frac{dY_t}{Y_t}}{\frac{dX_t}{X_t}} = \frac{\frac{dY_t}{Y_t}}{\frac{dX_t}{X_t}} = \frac{\Delta\%Y}{\Delta\%X}$$

$$\Delta\%Y = \beta_2 * \Delta\%X$$

The slope represents a constant elasticity, i.e., if X increases by 1%, then Y increases by β_2%.

ii. $\boldsymbol{Y_t = \beta_1 + \beta_2 X_t + u_t}$

$$\beta_2 = \frac{dY_t}{dX_t} = \frac{\Delta Y_t}{\Delta X_t}$$

$$\Delta Y_t = \beta_2 * \Delta X_t$$

The slope represents a direct change, i.e. if X increases by 1 unit, then Y increases by β_2 units.

iii. $\boldsymbol{\ln(Y_t) = \beta_1 + \beta_2 X_t + u_t}$

$$\ln Y_t - \beta_1 - \beta_2 X_t - u_t = 0$$

$$\frac{dY_t}{dX_t} = -\frac{\frac{dG}{dX_t}}{\frac{dG}{dY_t}} = -\frac{-\beta_2}{\frac{1}{Y_t}} = \beta_2 \, Y_t$$

$$\frac{dY_t}{dX_t} = \beta_2 \, Y_t$$

$$\beta_2 = \frac{dY_t}{dX_t} * \frac{1}{Y_t} = \frac{\frac{dY_t}{Y_t}}{\frac{dX_t}{1}} = \frac{\Delta\%Y/100}{\Delta X}$$

$$\Delta\%Y = \beta_2 * \Delta X * 100$$

The slope represents the semi-elasticity of Y with respect to X, i.e., if X increases by 1 unit, then Y increases by $100 * \beta_2\%$. For example if β_2=0.02, it means that if X increases by 1 unit, then Y increases 2%.

iv. $\boldsymbol{Y_t = \beta_1 + \beta_2 \ln X_t + u_t}$

$$\frac{dY_t}{dX_t} = \beta_2 * \frac{1}{X_t}$$

$$\beta_2 = \frac{dY_t}{dX_t} * \frac{X_t}{1} = \frac{\frac{dY_t}{1}}{\frac{dX_t}{X_t}} = \frac{\Delta Y}{\Delta\%X/100}$$

$$\Delta Y = \frac{\beta_2}{100} * \Delta\%X$$

In this case, if X increases by 1%, then Y increases by $0.01 \; \beta_2$ units. For example if β_2=200, it means that if X increases by 1%, then Y increases 2 units (200*0.01).

v. $\boldsymbol{Y_t = \beta_1 + \beta_2 \, 1/X_t + u_t}$

$$\frac{dY_t}{dX_t} = -\beta_2 * \frac{1}{X_t^2}$$

$$\beta_2 = -\frac{dY_t}{dX_t} * \frac{X_t^2}{1} = -\frac{\frac{dY_t}{1}}{\frac{dX_t}{X_t}} * X_t = -\frac{\Delta Y}{\Delta\%X/100} * X_t$$

$$\Delta Y = -\frac{\beta_2}{X_t * 100} * \Delta\%X$$

In this case, if X increases by 1%, then the units Y increases are equal to 0.01 β_2 times $(-1/X_t)$. For example if β_2=200, and $X_t = 200$ it means that if X increases by 1%, then Y decreases by 0.01 units [200*0.01*(-1/200) = - 0.01].

TABLE 6.6

Model	Equation	Slope $\left(=\frac{dY}{dX}\right)$	Elasticity $\left(=\frac{dY}{dX}\frac{X}{Y}\right)$
Linear	$Y = \beta_1 + \beta_2 X$	β_2	$\beta_2\left(\frac{X}{Y}\right)^*$
Log–linear	$\ln Y = \beta_1 + \beta_2 \ln X$	$\beta_2\left(\frac{Y}{X}\right)$	β_2
Log–lin	$\ln Y = \beta_1 + \beta_2 X$	$\beta_2 (Y)$	$\beta_2 (X)^*$
Lin–log	$Y = \beta_1 + \beta_2 \ln X$	$\beta_2\left(\frac{1}{X}\right)$	$\beta_2\left(\frac{1}{Y}\right)^*$
Reciprocal	$Y = \beta_1 + \beta_2\left(\frac{1}{X}\right)$	$-\beta_2\left(\frac{1}{X^2}\right)$	$-\beta_2\left(\frac{1}{XY}\right)^*$
Log reciprocal	$\ln Y = \beta_1 - \beta_2\left(\frac{1}{X}\right)$	$\beta_2\left(\frac{Y}{X^2}\right)$	$\beta_2\left(\frac{1}{X}\right)^*$

Note: * indicates that the elasticity is variable, depending on the value taken by *X* or *Y* or both. When no *X* and *Y* values are specified, in practice, very often these elasticities are measured at the mean values of these variables, namely, $\bar{X}$ and $\bar{Y}$.