Answers of Wooldridge 2.10

64 Parte I Análisis de regresión con ...

2.10 Sean $\hat{\beta}_0$ y $\hat{\beta}_1$ los estimadores de MCO del intercepto y de la pendiente, respectivamente, y sea $\bar{u}$ la media muestral de los errores (¡no de los residuales!).

i) Muestre que $\hat{\beta}_1$ se puede expresar como $\hat{\beta}_1 = \beta_1 + \sum_{i=1}^{n} w_i u_i$ donde $w_i = d_i/\mathrm{STC}_x$ y $d_i = x_i - \bar{x}$.

ii) Use el inciso i), así como $\sum_{i=1}^{n} w_i = 0$, para mostrar que $\hat{\beta}_1$ y $\bar{u}$ no están correlacionados. [*Sugerencia*: Se pide que demuestre que $\mathrm{E}[(\hat{\beta}_1 - \beta_1)\cdot \bar{u}] = 0$.]

iii) Muestre que $\hat{\beta}_0$ se puede expresar como $\hat{\beta}_0 = \beta_0 + \bar{u} - (\hat{\beta}_1 - \beta_1)\bar{x}$.

iv) Use los incisos ii) y iii) para mostrar que $\mathrm{Var}(\hat{\beta}_0) = \sigma^2/n + \sigma^2(\bar{x})^2/\mathrm{STC}_x$.

v) Simplifique la expresión del inciso iv) para llegar a la ecuación (2.58). [*Sugerencia*: $\mathrm{STC}_x/n = n^{-1}\sum_{i=1}^{n} x_i^2 - (\bar{x})^2$.]

... efecto de las horas invertidas en un curso de preparación para

i) Muestre que $\widehat{\beta_1}$ puede expresarse como $\widehat{\beta_1} = \beta_1 + \sum k_i u_i$ donde $k_i = \frac{x_i}{\sum x_i^2}$

$$\widehat{\beta_1} = \frac{\sum x_i Y_i}{\sum x_i^2} = \frac{\sum x_i(\beta_0 + X_i\beta_1 + u_i)}{\sum x_i^2} = \frac{\beta_0 \sum x_i + \beta_1 \sum x_i^2 + \sum x_i u_i}{\sum x_i^2}$$

$$\widehat{\beta_1} = \frac{\beta_1 \sum x_i^2}{\sum x_i^2} + \frac{\sum x_i u_i}{\sum x_i^2} = \beta_1 + \frac{\sum x_i u_i}{\sum x_i^2} = \beta_1 + \sum k_i u_i$$

$$\widehat{\beta_1} - \beta_1 = \sum k_i u_i$$

Some important properties

- $\sum x_i y_i = \sum x_i Y_i = \sum X_i y_i$ because:
  - $\sum \boldsymbol{x_i y_i} = \sum(X_i - \bar{X})(Y_i - \bar{Y}) = \sum(X_iY_i - X_i\bar{Y} - Y_i\bar{X} + \bar{X}\bar{Y}) = \sum X_iY_i - \sum X_i\bar{Y} - \sum Y_i\bar{X} + \sum \bar{X}\bar{Y} = \sum X_iY_i - N\bar{Y}\bar{X} - N\bar{Y}\bar{X} + N\bar{Y}\bar{X} = \sum \boldsymbol{X_iY_i} - \boldsymbol{N\bar{X}\bar{Y}}$
  - $\sum x_iY_i = \sum(X_i - \bar{X})(Y_i) = \sum(X_iY_i - Y_i\bar{X}) = \sum X_iY_i - \sum Y_i\bar{X} = \sum \boldsymbol{X_iY_i} - \boldsymbol{N\bar{X}\bar{Y}}$
  - $\sum X_iy_i = \sum(Y_i - \bar{Y})(X_i) = \sum(X_iY_i - X_i\bar{Y}) = \sum X_iY_i - \sum X_i\bar{Y} = \sum \boldsymbol{X_iY_i} - \boldsymbol{N\bar{X}\bar{Y}}$

$$\hat{\beta} = \frac{\sum x_iY_i}{\sum x_i^2} = \sum k_iY_i$$

$$\sum k_i = \sum \frac{x_i}{\sum x_i^2} = \frac{\sum x_i}{\sum x_i^2} = \frac{\sum(X_i - \bar{X})}{\sum x_i^2} = \frac{\sum(X_i) - N\bar{X}}{\sum x_i^2} = \frac{N\bar{X} - N\bar{X}}{\sum x_i^2} = \frac{0}{\sum x_i^2} = 0$$

$$\sum k_i X_i = \sum \frac{x_iX_i}{\sum x_i^2} = \frac{\sum x_i X_i}{\sum x_i^2}$$

- $\sum x_iX_i = \sum(X_i - \bar{X})(X_i) = \sum(X_i^2 - X_i\bar{X}) = \sum X_i^2 - \sum X_i\bar{X} = \sum \boldsymbol{X_i^2} - \boldsymbol{N\bar{X}^2}$

$$\sum k_i X_i = \frac{\sum \boldsymbol{X_i^2} - \boldsymbol{N\bar{X}^2}}{\sum x_i^2} = \frac{\sum \boldsymbol{X_i^2} - \boldsymbol{N\bar{X}^2}}{\sum(X_i - \bar{X})^2} = \frac{\sum \boldsymbol{X_i^2} - \boldsymbol{N\bar{X}^2}}{\sum X_i^2 - N\bar{X}^2} = 1$$

ii) Probar que $\boldsymbol{COV[\widehat{\beta_1}, \bar{u}] = 0}$

$$COV[\widehat{\beta_1}, \bar{u}] = COV[\widehat{\beta_1} - \beta_1, \bar{u}] = E[(\widehat{\beta_1} - \beta_1) * \bar{u}]$$

$$E[(\widehat{\beta_1} - \beta_1) * \bar{u}] = E\left[\left(\sum [k_i u_i]\right)\bar{u}\right] = E\left[\sum [k_i u_i \bar{u}]\right] = \sum [k_i E[u_i \bar{u}]] = \sum [k_i E[u_i \bar{u}]]$$

- $E[u_i \bar{u}] = \frac{1}{n} E[u_i * (u_1 + u_2 + \cdots + u_n)] = \frac{1}{n} E\left[u_i u_i + u_i \sum_{i \neq j}^{n} u_j\right] = E\left[\frac{u_i^2}{n}\right] = \frac{\sigma^2}{n}$

$$\sum [k_i E[u_i \bar{u}]] = \sum \left[k_i \frac{\sigma^2}{n}\right] = \sum [k_i] \frac{\sigma^2}{n} = 0 * \frac{\sigma^2}{n} = 0$$

$$E[(\widehat{\beta_1} - \beta_1) * \bar{u}] = 0$$

-----

iii) Mostrar que $\boldsymbol{\widehat{\beta}_0 = \beta_0 + \bar{u} + (\beta_1 - \widehat{\beta}_1)\bar{X}}$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

$$\bar{Y} = \beta_0 + \beta_1 \bar{X} + \bar{u}$$

$$\hat{\beta}_0 = (\beta_0 + \beta_1 \bar{X} + \bar{u}) - \hat{\beta}_1 \bar{X}$$

$$\hat{\beta}_0 = \beta_0 + \bar{u} + (\beta_1 - \hat{\beta}_1)\bar{X}$$

----

iv) **Mostrar que $\boldsymbol{var(\widehat{\beta}_0) = \sigma^2 \left(\frac{1}{n} + \frac{\bar{X}^2}{\sum x_i^2}\right)}$**

$$var(\hat{\beta}_0) = var(\beta_0 + \bar{u} + (\beta_1 - \hat{\beta}_1)\bar{X}) = var(\bar{u} + (\beta_1 - \hat{\beta}_1)\bar{X})$$

$$var(\hat{\beta}_0) = var(\bar{u}) + \bar{X}^2 var(\beta_1 - \hat{\beta}_1) + 2\bar{X} COV(\bar{u}, \beta_1 - \hat{\beta}_1)$$

- $COV(\bar{u}, \beta_1 - \hat{\beta}_1) = 0$

$$var(\hat{\beta}_0) = var(\bar{u}) + \bar{X}^2 var(\beta_1 - \hat{\beta}_1)$$

$$var(\hat{\beta}_0) = \frac{1}{n^2} var(u_1 + \cdots + u_n) + \bar{X}^2 var(\hat{\beta}_1)$$

$$var(\hat{\beta}_0) = \frac{1}{n^2}(n * var(u_i)) + \bar{X}^2 var(\hat{\beta}_1)$$

$$var(\hat{\beta}_0) = \frac{\sigma^2}{n} + \bar{X}^2 \frac{\sigma^2}{\sum x_i^2}$$

$$var(\hat{\beta}_0) = \sigma^2 \left(\frac{1}{n} + \frac{\bar{X}^2}{\sum x_i^2} \right)$$

v) **Mostrar que** $\boldsymbol{var(\widehat{\beta}_0)} = \sigma^2 \left(\frac{\sum [X_i^2]}{N \sum x_i^2} \right)$

$$var(\hat{\beta}_0) = \sigma^2 \left(\frac{1}{n} + \frac{(\sum X_i)^2 / n^2}{\sum x_i^2} \right)$$

$$var(\hat{\beta}_0) = \sigma^2 \left(\frac{\sum x_i^2}{n * \sum x_i^2} + \frac{n * (\sum X_i)^2 / n^2}{n * \sum x_i^2} \right)$$

$$var(\hat{\beta}_0) = \sigma^2 \left(\frac{\sum x_i^2 + (\sum X_i)^2 / n}{n * \sum x_i^2} \right)$$

$$var(\hat{\beta}_0) = \sigma^2 \left(\frac{[\sum (X_i^2) - \sum (X_i)^2 / n\ ] + (\sum X_i)^2 / n}{n * \sum x_i^2} \right)$$

$$var(\hat{\alpha}_{OLS}) = \sigma^2 \left(\frac{\sum [X_i^2]}{N \sum x_i^2} \right)$$