Ordinary Least Squares

Consider a model:

$$Y_i = \alpha + \beta X_i + u_i \quad i = 1,2, \dots , N$$

where the observations X_i are fixed and the disturbances are distributed independently and identically with mean zero and variance σ^2.

In OLS:

$$Min \sum \hat{u}_i^2 = \sum \left(Y_i - \hat{Y}_i\right)^2 = \sum \left(Y_i - \hat{\alpha} - \hat{\beta} X_i\right)^2$$

Normal Equations:

1. $\frac{\delta \sum \hat{u}_i^2}{\delta \hat{\alpha}} = -2 * \sum \left(Y_i - \hat{\alpha} - \hat{\beta} X_i\right)$

2. $\frac{\delta \sum \hat{u}_i^2}{\delta \hat{\beta}} = -2 * \sum \left(\left(Y_i - \hat{\alpha} - \hat{\beta} X_i\right) * X_i\right)$

From 1:

$$\sum \left(Y_i - \hat{\alpha} - \hat{\beta} X_i\right) = 0$$

$$\sum Y_i - \sum \hat{\alpha} - \sum \hat{\beta} X_i = 0$$

$$\sum Y_i - N\,\hat{\alpha} - \hat{\beta} \sum X_i = 0$$

$$N\,\hat{\alpha} = \sum Y_i - \hat{\beta} \sum X_i$$

$$\hat{\alpha} = \frac{\sum Y_i}{N} - \hat{\beta} \frac{\sum X_i}{N}$$

$$\hat{\alpha} = \bar{Y} - \hat{\beta} \bar{X}$$

From 2:

$$\sum \left(Y_i X_i - \hat{\alpha} X_i - \hat{\beta} X_i^2\right) = 0$$

$$\sum Y_i X_i - \sum \hat{\alpha} X_i - \sum \hat{\beta} X_i^2 = 0$$

$$\sum Y_i X_i - \hat{\alpha} \sum X_i - \hat{\beta} \sum X_i^2 = 0$$

$$\sum Y_i X_i - \left(\bar{Y} - \hat{\beta} \bar{X}\right)(\bar{X}\,N) - \hat{\beta} \sum X_i^2 = 0$$

$$\sum Y_i X_i - \left(N\,\bar{Y}\bar{X} - N\hat{\beta}\bar{X}^2\right) - \hat{\beta}\sum X_i^2 = 0$$

$$\sum Y_i X_i - N\,\bar{Y}\bar{X} + N\hat{\beta}\bar{X}^2 - \hat{\beta}\sum X_i^2 = 0$$

$$N\hat{\beta}\bar{X}^2 - \hat{\beta}\sum X_i^2 = -\sum Y_i X_i + N\,\bar{Y}\bar{X}$$

$$\hat{\beta}\left(\bar{X}^2 N - \sum X_i^2\right) = -\sum Y_i X_i + N\,\bar{Y}\bar{X}$$

$$\hat{\beta} = \frac{-\sum Y_i X_i + N\,\bar{Y}\bar{X}}{-\sum X_i^2 + N\bar{X}^2}$$

$$\hat{\beta} = \frac{\sum Y_i X_i - N\,\bar{Y}\bar{X}}{\sum X_i^2 - N\bar{X}^2}$$

$$\hat{\beta} = \frac{\frac{\sum Y_i X_i - N\,\bar{Y}\bar{X}}{N}}{\frac{\sum X_i^2 - N\bar{X}^2}{N}}$$

$$\hat{\beta} = \frac{\frac{\sum Y_i X_i}{N} - \bar{Y}\bar{X}}{\frac{\sum X_i^2}{N} - \bar{X}^2}$$

$$\hat{\beta} = \frac{Sample\ Cov(x,y)}{Sample\ Var(x)}$$

$$\hat{\beta} = \frac{\frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{N}}{\frac{\sum (X_i - \bar{X})^2}{N}}$$

$$\hat{\beta} = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2}$$

Note:

- $\sum (X_i - \bar{X})^2 = \sum X_i^2 - N\bar{X}^2$, because:
  - $\sum (X_i - \bar{X})^2 = \sum (X_i^2 - 2X_i\bar{X} + \bar{X}^2) = \sum X_i^2 - \sum 2X_i\bar{X} + \sum \bar{X}^2 = \sum X_i^2 - 2N\bar{X}\bar{X} + N\bar{X}^2 = \sum X_i^2 - 2N\bar{X}^2 + N\bar{X}^2 = \sum X_i^2 - N\bar{X}^2$
- $\sum (X_i - \bar{X})(Y_i - \bar{Y}) = \sum X_i Y_i - N\bar{X}\bar{Y}$, because:
  - $\sum (X_i - \bar{X})(Y_i - \bar{Y}) = \sum (X_i Y_i - X_i\bar{Y} - Y_i\bar{X} + \bar{X}\bar{Y}) = \sum X_i Y_i - \sum X_i\bar{Y} - \sum Y_i\bar{X} + \sum \bar{X}\bar{Y} = \sum X_i Y_i - N\bar{Y}\bar{X} - N\bar{Y}\bar{X} + N\bar{Y}\bar{X} = \sum X_i Y_i - N\bar{X}\bar{Y}$

$$\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}$$

$$\hat{\beta} = \frac{\sum x_i Y_i}{\sum x_i^2}$$

$$\hat{\beta} = \frac{\sum X_i y_i}{\sum x_i^2}$$

Note:

- $\sum x_i y_i = \sum x_i Y_i = \sum X_i y_i$ because:
    - $\sum \boldsymbol{x_i y_i} = \sum (X_i - \bar{X})(Y_i - \bar{Y}) = \sum (X_i Y_i - X_i \bar{Y} - Y_i \bar{X} + \bar{X}\bar{Y}) = \sum X_i Y_i - \sum X_i \bar{Y} - \sum Y_i \bar{X} + \sum \bar{X}\bar{Y} = \sum X_i Y_i - N\bar{Y}\bar{X} - N\bar{Y}\bar{X} + N\bar{Y}\bar{X} = \sum \boldsymbol{X_i Y_i} - \boldsymbol{N\bar{X}\bar{Y}}$
    - $\sum x_i Y_i = \sum (X_i - \bar{X})(Y_i) = \sum (X_i Y_i - Y_i \bar{X}) = \sum X_i Y_i - \sum Y_i \bar{X} = \sum \boldsymbol{X_i Y_i} - \boldsymbol{N\bar{X}\bar{Y}}$
    - $\sum X_i y_i = \sum (Y_i - \bar{Y})(X_i) = \sum (X_i Y_i - X_i \bar{Y}) = \sum X_i Y_i - \sum X_i \bar{Y} = \sum \boldsymbol{X_i Y_i} - \boldsymbol{N\bar{X}\bar{Y}}$

Proof that $\widehat{\boldsymbol{\beta}}_{OLS}$ is BLUE

a) **Linearity**

Prove that $\widehat{\boldsymbol{\beta}}_{OLS} = \sum \boldsymbol{k_i Y_i}$ where $\boldsymbol{k_i} = \frac{\boldsymbol{x_i}}{\sum \boldsymbol{x_i^2}}$

$$\hat{\beta} = \frac{\sum x_i Y_i}{\sum x_i^2} = \sum k_i Y_i$$

b) **Proof of unbiasedness**

Prove that $\sum \boldsymbol{k_i} = \mathbf{0}$ and $\sum \boldsymbol{k_i X_i} = \mathbf{1}$

$$\sum k_i = \sum \frac{x_i}{\sum x_i^2} = \frac{\sum x_i}{\sum x_i^2} = \frac{\sum (X_i - \bar{X})}{\sum x_i^2} = \frac{\sum (X_i) - N\bar{X}}{\sum x_i^2} = \frac{N\bar{X} - N\bar{X}}{\sum x_i^2} = \frac{0}{\sum x_i^2} = 0$$

$$\sum k_i X_i = \sum \frac{x_i X_i}{\sum x_i^2} = \frac{\sum x_i X_i}{\sum x_i^2}$$

- $\sum x_i X_i = \sum (X_i - \bar{X})(X_i) = \sum (X_i^2 - X_i \bar{X}) = \sum X_i^2 - \sum X_i \bar{X} = \sum \boldsymbol{X_i^2} - \boldsymbol{N\bar{X}^2}$

$$\sum k_i X_i = \frac{\sum \boldsymbol{X_i^2} - \boldsymbol{N\bar{X}^2}}{\sum x_i^2} = \frac{\sum \boldsymbol{X_i^2} - \boldsymbol{N\bar{X}^2}}{\sum (X_i - \bar{X})^2} = \frac{\sum \boldsymbol{X_i^2} - \boldsymbol{N\bar{X}^2}}{\sum X_i^2 - N\bar{X}^2} = 1$$

c) **Continuing proof of unbiasedness**

Prove that any other linear estimator of $\boldsymbol{\beta}$, say $\widetilde{\boldsymbol{\beta}} = \sum \boldsymbol{m_i Y_i}$, must satisfy $\sum \boldsymbol{m_i} = 0$ and $\sum \boldsymbol{m_i X_i} = 1$ to be unbiased for $\boldsymbol{\alpha}$.

Unbiased means: $E(\tilde{\beta}) = \beta$

If $\tilde{\beta} = \sum m_i Y_i$

$$E[\tilde{\beta}] = E\left[\sum m_i Y_i\right]$$

- $Y_i = \alpha + \beta X_i + u_i$

$$E(\tilde{\beta}) = E\left[\sum m_i(\alpha + \beta X_i + u_i)\right]$$

$$E(\tilde{\beta}) = E\left[\sum \alpha m_i + \beta X_i m_i + u_i m_i\right]$$

$$E(\tilde{\beta}) = E\left[\alpha \sum m_i + \beta \sum X_i m_i + \sum u_i m_i\right]$$

If $\sum m_i = 0$, $\sum m_i X_i = 1$:

$$E(\tilde{\beta}) = E\left[\beta + \sum u_i m_i\right]$$

$$E(\tilde{\beta}) = E[\beta] + E\left[\sum u_i m_i\right]$$

Since m_i is a function of X_i which is fixed, then m_i is not stochastic as well and therefore can be treated as a constant.

$$E(\tilde{\beta}) = E[\beta] + \sum m_i E[u_i]$$

- Since $E[u_i] = 0$ by assumption:

$$E(\tilde{\beta}) = E[\beta]$$

$$E(\tilde{\beta}) = \beta$$

Then, the only way $E(\tilde{\beta}) = \beta$ is if $\sum m_i = 0$and $\sum m_i X_i = 1$.

d) **Proof of Minimum Variance**

Consider $\tilde{\boldsymbol{\beta}}$and $\hat{\boldsymbol{\beta}}_{OLS}$ to be unbiased estimatorsof $\boldsymbol{\beta}$, i.e.

$$\tilde{\boldsymbol{\beta}} = \sum \boldsymbol{m_i Y_i} \sum \boldsymbol{m_i} = \mathbf{0} \sum \boldsymbol{m_i X_i} = \mathbf{1}$$

$$\hat{\boldsymbol{\beta}}_{OLS} = \sum \boldsymbol{k_i Y_i} \sum \boldsymbol{k_i} = \mathbf{0} \sum \boldsymbol{k_i X_i} = \mathbf{1}$$

Let $\boldsymbol{m_i} = \boldsymbol{k_i} + \boldsymbol{f_i}$, where $\boldsymbol{k_i}$ is from OLS and $\boldsymbol{f_i}$ is a fixed of $\boldsymbol{X_i}$.

d1) Show that $\sum \boldsymbol{f_i} = \mathbf{0}$

$$m_i = k_i + f_i$$

$$f_i = m_i - k_i$$

$$\sum f_i = \sum m_i - \sum k_i = 0 - 0 = 0$$

Since $\hat{\beta}_{OLS}$ and $\tilde{\beta}$ are unbiased estimators of β: $\sum m_i = 0$ and $\sum k_i = 0$

d2) Show that$\sum \boldsymbol{f_i X_i} = \mathbf{0}$

$$f_i = m_i - k_i$$

$$f_i x_i = (m_i - k_i) * X_i$$

$$f_i x_i = m_i X_i - k_i X_i$$

$$\sum f_i x_i = \sum m_i X - \sum k_i X_i = 1 - 1 = 0$$

Since $\hat{\beta}_{OLS}$ and $\tilde{\beta}$ are unbiased estimators of β: $\sum m_i X_i = 1$ and$\sum k_i X_i = 1$

e) Continuing Proof of Minimum Variance

Prove that $\boldsymbol{var(\tilde{\beta}) = \sigma^2 \sum m_i^2 = var(\hat{\beta}_{OLS}) + \sigma^2 \sum f_i^2}$

Remember that if $\tilde{\beta}$ is an unbiased estimator of β:

$$\tilde{\beta} = \sum m_i Y_i \sum m_i = 0 \sum m_i X_i = 1$$

We know that for any unbiased estimator of β:

$$var(\tilde{\beta}) = E[\tilde{\beta} - \beta]^2$$

From 3c, we know that:

- $E(\tilde{\beta}) = E[\beta] + E[\sum u_i m_i]$
- $E(\tilde{\beta} - \beta) = E[\sum u_i m_i]$
- $E(\tilde{\beta} - \beta)^2 = E[\sum u_i m_i]^2$

$$var(\tilde{\beta}) = E\left[\sum u_i m_i\right]^2$$

$$var(\tilde{\beta}) = E[u_1 m_1 + u_2 m_2 + \cdots u_N m_N]^2$$

$$var(\tilde{\beta}) = E[u_1^2 m_1^2 + u_2^2 m_2^2 + \cdots + u_N^2 m_N^2 + 2u_1 m_1 u_2 m_2 + \cdots 2u_{N-1} m_{N-1} u_N m_N]$$

Since

- $m_i$is a function of X_i which is fixed, then m_i is not stochastic as well and therefore can be treated as a constant.
- by assumption $E[u_i u_j] = 0, \forall\ i \neq j$
- by assumption $E[u_i^2] = \sigma^2$

$$var(\tilde{\beta}) = E[u_1^2 m_1^2 + u_2^2 m_2^2 + \cdots + u_N^2 m_N^2]$$

$$var(\tilde{\beta}) = m_1^2 E[u_1^2] + m_2^2 E[u_2^2] + \cdots + m_N^2 E[u_N^2]$$

$$var(\tilde{\beta}) = m_1^2 \sigma^2 + m_2^2 \sigma^2 + \cdots + m_N^2 \sigma^2$$

$$var(\tilde{\beta}) = \sigma^2 \sum m_i^2$$

Since the estimator of OLS is also unbiased, and the assumptions made here also hold:

$$var(\hat{\beta}_{OLS}) = \sigma^2 \sum k_i^2$$

Note that

$$\sum[k_i^2] = \sum\left[\left(\frac{x_i}{\sum x_i^2}\right)^2\right] = \sum\frac{x_i^2}{(\sum x_i^2)^2} = \frac{1}{(\sum x_i^2)^2} * \sum x_i^2 = \frac{1}{\sum x_i^2}$$

Thus:

$$var(\hat{\beta}_{OLS}) = \frac{\sigma^2}{\sum x_i^2}$$

$\boldsymbol{m_i = k_i + f_i,}$

Another way to express $var(\tilde{\beta})$ comes from doing the following:

$$var(\tilde{\beta}) = \sigma^2 \sum m_i^2$$

$$var(\tilde{\beta}) = \sigma^2 \sum (m_i - k_i + k_i)^2$$

$$var(\tilde{\beta}) = \sigma^2 \sum \left((b_i - k_i) + (k_i)\right)^2$$

$$var(\tilde{\beta}) = \sigma^2 \left[\sum (m_i - k_i)^2 + \sum k_i^2 + 2\sum (m_i - k_i) * (k_i)\right]$$

Note that:

$\sum(m_i - k_i) * (k_i) = 0$, because:

$$\sum (m_i - k_i) * (k_i) = \sum (m_i k_i) - \sum k_i^2$$

$$\sum (m_i - k_i) * (k_i) = \sum\left(m_i \frac{x_i}{\sum x_i^2}\right) - \frac{1}{\sum x_i^2}$$

$$\sum (m_i - k_i) * (k_i) = \frac{\sum m_i x_i}{\sum x_i^2} - \frac{1}{\sum x_i^2}$$

$$\sum (m_i - k_i) * (k_i) = \frac{1}{\sum x_i^2}\left(\sum [m_i x_i] - 1\right)$$

$$\sum (m_i - k_i) * (k_i) = \frac{1}{\sum x_i^2}\left(\sum [m_i (X_i - \bar{X})] - 1\right)$$

$$\sum (m_i - k_i) * (k_i) = \frac{1}{\sum x_i^2}\left(\sum [m_i X_i] - \sum [m_i \bar{X}] - 1\right)$$

$$\sum (m_i - k_i) * (k_i) = \frac{1}{\sum x_i^2}\left(\sum [m_i X_i] - \bar{X}\sum [m_i] - 1\right)$$

- $\sum m_i = 0$
- $\sum m_i X_i = 1$

$$\sum (m_i - k_i) * (k_i) = \frac{1}{\sum x_i^2}(1 - 0 - 1)$$

$$\sum (m_i - k_i) * (k_i) = 0$$

$$var(\tilde{\beta}) = \sigma^2 \left[\sum (m_i - k_i)^2 + \sum k_i^2\right]$$

$$var(\tilde{\beta}) = \sigma^2 \sum (m_i - k_i)^2 + \sigma^2 \sum k_i^2$$

Since $var(\hat{\beta}_{OLS}) = \sigma^2 \sum k_i^2$

$$var(\tilde{\beta}) = \sigma^2 \sum (m_i - k_i)^2 + var(\hat{\beta}_{OLS})$$

Since $m_i = k_i + f_i$, $m_i - k_i = f_i$

$$var(\tilde{\beta}) = \sigma^2 \sum {f_i}^2 + var(\hat{\beta}_{OLS})$$

Thus,

$$var(\tilde{\beta}) \geq var(\hat{\beta}_{OLS})$$

Proof that $\hat{\alpha}_{OLS}$ is BLUE

a) Linearity

Prove that $\hat{\alpha}_{OLS} = \sum \lambda_i Y_i$ where $\lambda_i = \frac{1}{n} - \bar{X}w_i$ and $w_i = \frac{x_i}{\sum x_i^2}$

That is, we need to prove $\hat{\alpha}_{OLS} = \sum \left(\frac{1}{N} - \bar{X}w_i\right) Y_i = \sum \left(\frac{1}{n} - \bar{X}\left(\frac{x_i}{\sum x_i^2}\right)\right)(Y_i)$

$$\hat{\alpha}_{OLS} = \sum \left(\frac{1}{N} - \bar{X}\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}\right)\right)(Y_i)$$

$$\hat{\alpha}_{OLS} = \sum \left(\frac{Y_i}{N} - \bar{X}Y_i\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}\right)\right)$$

$$\hat{\alpha}_{OLS} = \sum \left(\frac{Y_i}{N}\right) - \sum \left(\bar{X}Y_i\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}\right)\right)$$

$$\hat{\alpha}_{OLS} = \bar{Y} - \sum \left(\bar{X}Y_i\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}\right)\right)$$

Note that $\sum (X_i - \bar{X})^2$ is a constant. Therefore:

$$\hat{\alpha}_{OLS} = \bar{Y} - \bar{X}\frac{\sum \left(Y_i(X_i - \bar{X})\right)}{\sum (X_i - \bar{X})^2}$$

$$\hat{\alpha}_{OLS} = \bar{Y} - \bar{X}\frac{\sum (Y_iX_i - Y_i\bar{X})}{\sum (X_i - \bar{X})^2}$$

$$\hat{\alpha}_{OLS} = \bar{Y} - \bar{X}\frac{\sum (Y_iX_i) - \bar{X}\sum (Y_i)}{\sum (X_i - \bar{X})^2}$$

$$\hat{\alpha}_{OLS} = \bar{Y} - \bar{X}\frac{\sum (Y_iX_i) - N\bar{X}\bar{Y}}{\sum (X_i - \bar{X})^2}$$

- Remember: $\sum (X_i - \bar{X})^2 = \sum X_i^2 - N\bar{X}^2$

$$\hat{\alpha}_{OLS} = \bar{Y} - \bar{X}\frac{\sum (Y_iX_i) - N\bar{X}\bar{Y}}{\sum X_i^2 - N\bar{X}^2}$$

Since $\hat{\beta} = \frac{\sum Y_iX_i - N\bar{Y}\bar{X}}{\sum X_i^2 - N\bar{X}^2}$, then:

$$\hat{\alpha}_{OLS} = \bar{Y} - \bar{X}\hat{\beta}$$

b) Proof of unbiasedness

Show that $\sum \boldsymbol{\lambda_i} = \mathbf{1}$ and $\sum \boldsymbol{\lambda_i X_i} = \mathbf{0}$

- $\hat{\alpha}_{OLS} = \sum \lambda_i Y_i$ where $\lambda_i = \frac{1}{N} - \bar{X} w_i$ and $w_i = \frac{x_i}{\sum x_i^2}$
- $\hat{\alpha}_{OLS} = \sum \left(\frac{1}{N} - \bar{X}\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}\right)\right)(Y_i) = \bar{Y} - \bar{X}\hat{\beta}$

I.We need to show $\sum \lambda_i = 1$

$$\lambda_i = \frac{1}{N} - \bar{X}\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}\right)$$

$$\sum \lambda_i = \left[\sum \frac{1}{N} - \bar{X}\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}\right)\right]$$

$$\sum \lambda_i = \sum \frac{1}{N} - \sum \bar{X}\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}\right)$$

$$\sum \lambda_i = \frac{N}{N} - \frac{\bar{X}}{\sum (X_i - \bar{X})^2} \sum (X_i - \bar{X})$$

$$\sum \lambda_i = 1 - \frac{\bar{X}}{\sum (X_i - \bar{X})^2} * 0$$

$$\sum \lambda_i = 1$$

II.We need to show $\sum \lambda_i X_i = 0$

$$\lambda_i X_i = \left(\frac{1}{N} - \bar{X}\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}\right)\right) * X_i$$

$$\sum \lambda_i X_i = \sum \left(\frac{X_i}{N} - X_i \bar{X}\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}\right)\right)$$

$$\sum \lambda_i X_i = \sum \left(\frac{X_i}{N}\right) - \sum \left(X_i \bar{X}\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}\right)\right)$$

$$\sum \lambda_i X_i = \bar{X} - \frac{\bar{X}}{\sum (X_i - \bar{X})^2} \sum \big(X_i (X_i - \bar{X})\big)$$

$$\sum \lambda_i X_i = \bar{X} - \bar{X}\frac{\sum\big(X_i(X_i - \bar{X})\big)}{\sum(X_i - \bar{X})^2}$$

$$\sum \lambda_i X_i = \bar{X} - \bar{X}\frac{\sum(X_i^2 - X_i\bar{X})}{\sum(X_i - \bar{X})^2}$$

Remember: $\sum(X_i - \bar{X})^2 = \sum X_i^2 - N\bar{X}^2$

$$\sum \lambda_i X_i = \bar{X} - \bar{X}\frac{\sum(X_i^2 - X_i\bar{X})}{\sum X_i^2 - N\bar{X}^2}$$

$$\sum \lambda_i X_i = \bar{X} - \bar{X}\frac{\sum X_i^2 - N\bar{X}\bar{X}}{\sum X_i^2 - N\bar{X}^2}$$

$$\sum \lambda_i X_i = \bar{X} - \bar{X}\frac{\sum X_i^2 - N\bar{X}^2}{\sum X_i^2 - N\bar{X}^2}$$

$$\sum \lambda_i X_i = \bar{X} - \bar{X}$$

$$\sum \lambda_i X_i = 0$$

c) Continuing proof of unbiasedness

Prove that any other linear estimator of α, say $\tilde{\alpha} = \sum b_i Y_i$, must satisfy $\sum b_i = 1$ and $\sum b_i X_i = 0$ to be unbiased for α.

Unbiased means: $E(\tilde{\alpha}) = \alpha$

If $\tilde{\alpha} = \sum b_i Y_i$

$$E(\tilde{\alpha}) = E\left[\sum b_i Y_i\right]$$

- $Y_i = \alpha + \beta X_i + u_i$

$$E(\tilde{\alpha}) = E\left[\sum b_i(\alpha + \beta X_i + u_i)_i\right]$$

$$E(\tilde{\alpha}) = E\left[\sum \alpha b_i + \beta X_i b_i + u_i b_i\right]$$

$$E(\tilde{\alpha}) = E\left[\sum \alpha b_i + \sum \beta X_i b_i + \sum u_i b_i\right]$$

$$E(\tilde{\alpha}) = E\left[\alpha \sum b_i + \beta \sum X_i b_i + \sum u_i b_i\right]$$

If $\sum b_i = 1$, $\sum b_i X_i = 0$:

$$E(\tilde{\alpha}) = E\left[\alpha + \sum u_i b_i\right]$$

$$E(\tilde{\alpha}) = E[\alpha] + E\left[\sum u_i b_i\right]$$

$$E(\tilde{\alpha}) = E[\alpha] + \sum E[u_i b_i]$$

Since b_i is a function of X_i which is fixed, then b_i is not stochastic as well and therefore can be treated as a constant.

$$E(\tilde{\alpha}) = E[\alpha] + \sum b_i\, E[u_i]$$

- Since $E[u_i] = 0$ by assumption:

$$E(\tilde{\alpha}) = E[\alpha]$$

$$E(\tilde{\alpha}) = \alpha$$

Then, the only way $E(\tilde{\alpha}) = \alpha$ is if $\sum b_i = 1$and $\sum b_i X_i = 0$.

d) Proof of Minimum Variance

Let $\boldsymbol{b_i = \lambda_i + f_i}$. Show that $\boldsymbol{\sum f_i = 0}$ and $\boldsymbol{\sum f_i X_i = 0}$

- $\hat{\alpha}_{OLS} = \sum \lambda_i Y_i$ where $\lambda_i = \frac{1}{N} - \bar{X} w_i$ and $w_i = \frac{x_i}{\sum x_i^2}$
- $\hat{\alpha}_{OLS} = \sum \left(\frac{1}{N} - \bar{X}\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}\right)\right)(Y_i) = \bar{Y} - \bar{X}\hat{\beta}$
- $\sum \lambda_i = 1$ and $\sum \lambda_i X_i = 0$
- Considering $\tilde{\alpha}$ is an unbiased estimator of α, i.e.
    - $\tilde{\alpha} = \sum b_i Y_i \sum b_i = 1 \sum b_i X_i = 0$

I. We need to show $\sum f_i = 0$

$$b_i = \lambda_i + f_i$$

$$f_i = b_i - \lambda_i$$

$$\sum f_i = \sum b_i - \sum \lambda_i$$

Since $\hat{\alpha}_{OLS}$ and $\tilde{\alpha}$ are unbiased estimators of α: $\sum b_i = 1$ and $\sum \lambda_i = 1$

$$\sum f_i = 1 - 1$$

$$\sum f_i = 0$$

II. We need to show $\sum f_i X_i = 0$

$$f_i = b_i - \lambda_i$$

$$f_i x_i = (b_i - \lambda_i) * X_i$$

$$f_i x_i = b_i X_i - \lambda_i X_i$$

$$\sum f_i x_i = \sum b_i X - \sum \lambda_i X_i$$

Since $\hat{\alpha}_{OLS}$ and $\tilde{\alpha}$ are unbiased estimators of α: $\sum b_i X_i = 0$ and $\sum \lambda_i X_i = 0$

$$\sum f_i x_i = 0$$

e) Continuing Proof of Minimum Variance

Prove that $\boldsymbol{var(\tilde{\alpha}) = \sigma^2 \sum b_i^2 = var(\hat{\alpha}_{OLS}) + \sigma^2 \sum f_i^2}$

- Remember that if $\tilde{\alpha}$ is an unbiased estimator of α:
  - $\tilde{\alpha} = \sum b_i Y_i \sum b_i = 1 \sum b_i X_i = 0$

We know that for any unbiased estimator of α:

$$var(\tilde{\alpha}) = E\big[\tilde{\alpha} - E[\tilde{\alpha}]\big]^2$$

$$var(\tilde{\alpha}) = E[\tilde{\alpha} - \alpha]^2$$

From exercise 3c, we know that:

- $E(\tilde{\alpha}) = E[\alpha] + E[\sum u_i b_i]$
- $E(\tilde{\alpha}) = \alpha + E[\sum u_i b_i]$
- $E(\tilde{\alpha} - \alpha) = E[\sum u_i b_i]$
- $E(\tilde{\alpha} - \alpha)^2 = E[\sum u_i b_i]^2$

$$var(\tilde{\alpha}) = E\left[\sum u_i b_i\right]^2$$

$$var(\tilde{\alpha}) = E[u_1 b_1 + u_2 b_2 + \cdots u_N b_N]^2$$

$$var(\tilde{\alpha}) = E[(u_1 b_1 + u_2 b_2 + \cdots + u_N b_N) * (u_1 b_1 + u_2 b_2 + \cdots + u_N b_N)]$$

$$var(\tilde{\alpha}) = E[u_1^2 b_1^2 + u_2^2 b_2^2 + \cdots + u_N^2 b_N^2 + 2u_1 b_1 u_2 b_2 + \cdots 2u_{N-1} b_{N-1} u_N b_N]$$

Since

- $b_i$is a function of X_i which is fixed, then b_i is not stochastic as well and therefore can be treated as a constant.
- by assumption $E[u_i u_j] = 0, \forall\, i \neq j$
- by assumption $E[u_i^2] = \sigma^2$

$$var(\tilde{\alpha}) = E[u_1^2 b_1^2 + u_2^2 b_2^2 + \cdots + u_N^2 b_N^2]$$

$$var(\tilde{\alpha}) = b_1^2 E[u_1^2] + b_2^2 E[u_2^2] + \cdots + b_N^2 E[u_N^2]$$

$$var(\tilde{\alpha}) = b_1^2 \sigma^2 + b_2^2 \sigma^2 + \cdots + b_N^2 \sigma^2$$

$$var(\tilde{\alpha}) = \sigma^2 \sum b_i^2$$

Since the estimator of OLS is also unbiased, and the assumptions made here also hold:

$$var(\hat{\alpha}_{OLS}) = \sigma^2 \sum \lambda_i^2$$

Also, note that:

$$\lambda_i = \frac{1}{N} - \bar{X}\left(\frac{x_i}{\sum x_i^2}\right)$$

$$\sum \lambda_i^2 = \sum \left(\frac{1}{N} - \bar{X}\left(\frac{x_i}{\sum x_i^2}\right)\right)^2$$

$$\sum \lambda_i^2 = \sum \left(\frac{1}{N}\right)^2 + \sum \left(\bar{X}\left(\frac{x_i}{\sum x_i^2}\right)\right)^2 - 2\sum \left(\frac{1}{N}\right)\bar{X}\left(\frac{x_i}{\sum x_i^2}\right)$$

$$\sum \lambda_i^2 = \frac{1}{N} + \frac{\bar{X}^2}{(\sum x_i^2)^2}\sum x_i^2 - 2\left(\frac{\bar{X}}{N\ \sum x_i^2}\right)\sum (x_i)$$

$$\sum \lambda_i^2 = \frac{1}{N} + \frac{\bar{X}^2}{\sum x_i^2}$$

$$\sum \lambda_i^2 = \frac{\sum x_i^2 + N\bar{X}^2}{N \sum x_i^2}$$

- Remember: $\sum (X_i - \bar{X})^2 = \sum x_i^2 = \sum X_i^2 - N\bar{X}^2$

$$\sum \lambda_i^2 = \frac{\sum X_i^2 - N\bar{X}^2 + N\bar{X}^2}{N\sum x_i^2}$$

$$\sum \lambda_i^2 = \frac{\sum X_i^2}{N\sum x_i^2}$$

$$var(\hat{\alpha}_{OLS}) = \sigma^2\left(\frac{\sum X_i^2}{N\sum x_i^2}\right)$$

Another way to express $var(\tilde{\alpha})$ comes from doing the following:

$$var(\tilde{\alpha}) = \sigma^2 \sum b_i^2$$

$$var(\tilde{\alpha}) = \sigma^2 \sum (b_i - \lambda_i + \lambda_i)^2$$

$$var(\tilde{\alpha}) = \sigma^2 \sum \big((b_i - \lambda_i) + (\lambda_i)\big)^2$$

$$var(\tilde{\alpha}) = \sigma^2 \left[\sum (b_i - \lambda_i)^2 + \sum \lambda_i^2 + 2\sum (b_i - \lambda_i) * (\lambda_i)\right]$$

Note that:

$\sum(b_i - \lambda_i) * (\lambda_i) = 0$, because:

$$\sum (b_i - \lambda_i) * (\lambda_i) = \sum (b_i\lambda_i) - \sum \lambda_i^2$$

$$\sum (b_i - \lambda_i) * (\lambda_i) = \sum \left(b_i\left(\frac{1}{N} - \bar{X}\left(\frac{x_i}{\sum x_i^2}\right)\right)\right) - \frac{\sum X_i^2}{N\sum x_i^2}$$

$$\sum (b_i - \lambda_i) * (\lambda_i) = \sum \left(\frac{b_i}{N} - \left(\frac{\bar{X}b_i x_i}{\sum x_i^2}\right)\right) - \frac{\sum X_i^2}{N\sum x_i^2}$$

$$\sum (b_i - \lambda_i) * (\lambda_i) = \sum \left(\frac{b_i}{N}\right) - \sum \left(\frac{\bar{X}b_i x_i}{\sum x_i^2}\right) - \frac{\sum X_i^2}{N\sum x_i^2}$$

$$\sum (b_i - \lambda_i) * (\lambda_i) = \frac{1}{N} - \frac{\bar{X}}{\sum x_i^2}\sum \big(b_i(X_i - \bar{X})\big) - \frac{\sum X_i^2}{N\sum x_i^2}$$

$$\sum (b_i - \lambda_i) * (\lambda_i) = \frac{1}{N} - \frac{\bar{X}}{\sum x_i^2}\left[\sum (b_i X_i - b_i\bar{X})\right] - \frac{\sum X_i^2}{N\sum x_i^2}$$

$$\sum(b_i-\lambda_i)*(\lambda_i)=\frac{1}{N}-\frac{\bar{X}}{\sum x_i^2}\left[\sum(b_iX_i)-\sum(b_i\bar{X})\right]-\frac{\sum X_i^2}{N\sum x_i^2}$$

$$\sum(b_i-\lambda_i)*(\lambda_i)=\frac{1}{N}-\frac{\bar{X}}{\sum x_i^2}\left[0-\sum(b_i\bar{X})\right]-\frac{\sum X_i^2}{N\sum x_i^2}$$

$$\sum(b_i-\lambda_i)*(\lambda_i)=\frac{1}{N}+\frac{\bar{X}^2}{\sum x_i^2}\left[\sum(b_i)\right]-\frac{\sum X_i^2}{N\sum x_i^2}$$

$$\sum(b_i-\lambda_i)*(\lambda_i)=\frac{1}{N}+\frac{\bar{X}^2}{\sum x_i^2}-\frac{\sum X_i^2}{N\sum x_i^2}$$

$$\sum(b_i-\lambda_i)*(\lambda_i)=\frac{\sum x_i^2+N\bar{X}^2}{N\sum x_i^2}-\frac{\sum X_i^2}{N\sum x_i^2}$$

$$\sum(b_i-\lambda_i)*(\lambda_i)=\frac{\sum x_i^2+N\bar{X}^2}{N\sum x_i^2}-\frac{\sum X_i^2}{N\sum x_i^2}$$

Since $\sum x_i^2=\sum X_i^2-N\bar{X}^2$

$$\sum(b_i-\lambda_i)*(\lambda_i)=\frac{\sum X_i^2-N\bar{X}^2+N\bar{X}^2}{N\sum x_i^2}-\frac{\sum X_i^2}{N\sum x_i^2}$$

$$\sum(b_i-\lambda_i)*(\lambda_i)=\frac{\sum X_i^2}{N\sum x_i^2}-\frac{\sum X_i^2}{N\sum x_i^2}$$

$$\sum(b_i-\lambda_i)*(\lambda_i)=0$$

$$var(\tilde{\alpha})=\sigma^2\left[\sum(b_i-\lambda_i)^2+\sum\lambda_i^2\right]$$

$$var(\tilde{\alpha})=\sigma^2\sum(b_i-\lambda_i)^2+\sigma^2\sum\lambda_i^2$$

Since $var(\hat{\alpha}_{OLS})=\sigma^2\sum\lambda_i^2$and $b_i=\lambda_i+f_i$.

$$var(\tilde{\alpha})=\sigma^2\sum f_i^2+var(\hat{\alpha}_{OLS})$$

Thus:

$$var(\tilde{\alpha})>var(\hat{\alpha}_{OLS})$$