**1. Bernoulli**

1. We know our PMF is:

$$f(y_i|\theta) = \theta^{y_i}(1-\theta)^{1-y_i}$$

2. We obtain the Joint Density Function:

$$f(y_1, y_2, \dots, y_T|\theta) = L(\theta|y_1, y_2, \dots, y_T) = L(\theta|\mathbf{y}) = \prod_{i=1}^{T}[\theta^{y_i}(1-\theta)^{1-y_i}]$$

3. Take the natural logarithm of the Joint Density Function (monotonic transformation):

$$\ln L(\theta|y) = \sum_{i=1}^{T}[\ln(\theta^{y_i}(1-\theta)^{1-y_i})]$$

$$\ln L(\theta|y) = \sum_{i=1}^{T}[\ln(\theta^{y_i}) + \ln(1-\theta)^{1-y_i}]$$

$$\ln L(\theta|y) = \sum_{i=1}^{T}[\ln(\theta^{y_i})] + \sum_{i=1}^{T}[\ln(1-\theta)^{1-y_i}]$$

$$\ln L(\theta|y) = \ln(\theta)\sum_{i=1}^{T}[y_i] + \ln(1-\theta)\sum_{i=1}^{T}[1-y_i]$$

4. Maximize the expression obtained in 3 and derive the estimated parameter of interest.

**Note:**

- $\frac{\partial}{\partial x}\ \ln(u) = \frac{1}{u}\frac{\partial u}{\partial x}$

$$\frac{d\ln L(\theta|y)}{d\theta} = \frac{\sum(y_i)}{\theta} - \frac{\sum(1-y_i)}{1-\theta}$$

**FOC:**

$$\frac{d\ln L(\theta|y)}{d\theta} = \frac{\sum(y_i)}{\tilde{\theta}} - \frac{\sum(1-y_i)}{1-\tilde{\theta}} = 0$$

$$\frac{\sum(y_i)}{\tilde{\theta}} = \frac{\sum(1-y_i)}{1-\tilde{\theta}}$$

$$\frac{1-\tilde{\theta}}{\tilde{\theta}} = \frac{\sum(1-y_i)}{\sum(y_i)}$$

$$\frac{1}{\tilde{\theta}} - 1 = \frac{T-\sum(y_i)}{\sum(y_i)}$$

$$\frac{1}{\tilde{\theta}} = \frac{T}{\sum(y_i)}$$

$$\tilde{\theta} = \frac{\sum(y_i)}{T} = \bar{Y} \quad ; \quad T\ Observations, 1\ Trial$$

**2. Poisson**

1. We know our density function is:

$$f(x_i;\lambda) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

2. We obtain the Joint Density Function:

$$L(\lambda|\boldsymbol{x}) = f\left(x_{0,}, x_1, \dots, x_n; \lambda\right) = \prod \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

3. Take the natural logarithm of the Joint Density Function (monotonic transformation):

$$\ln[L(\lambda|\boldsymbol{x})] = \ln\left[\prod \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right] = \sum \ln\left(\frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right)$$

$$\ln[L(\lambda|\boldsymbol{x})] = \sum \left[\ln\left(e^{-\lambda}\right) + \ln(\lambda^{x_i}) - \ln(x_i!)\right]$$

$$\ln[L(\lambda|\boldsymbol{x})] = \sum [-\lambda + x_i \ln(\lambda) - \ln(x_i!)]$$

$$\ln[L(\lambda|\boldsymbol{x})] = -\lambda N + \ln(\lambda) \sum x_i - \sum \ln(x_i!)$$

4. Maximize the expression obtained in 3 and derive the estimated parameter of interest.

$$\frac{d \ln L}{d\lambda} = -N + \frac{1}{\lambda}\sum x_i$$

**FOC:**

$$-N + \frac{1}{\tilde{\lambda}}\sum x_i = 0$$

$$\frac{1}{\tilde{\lambda}} = \frac{N}{\sum x_i}$$

$$\tilde{\lambda} = \frac{\sum x_i}{N} = \bar{x}; \quad N\ Observations$$

**3. Binomial**

1. We know our density function is:

$$f(y|\theta) = \left(\frac{n!}{y!\,(n-y)!}\right)\theta^{y}(1-\theta)^{n-y}$$

2. We obtain the Joint Density Function:

$$f(y_1, y_2, \dots, y_T|\theta) = L(\theta|y_1, y_2, \dots, y_T) = L(\theta|\mathbf{y}) = \prod_{i=1}^{T}\left[\left(\frac{n!}{y_i!\,(n-y_i)!}\right)\theta^{y_i}(1-\theta)^{n-y_i}\right]$$

3. Take the natural logarithm of the Joint Density Function (monotonic transformation):

$$\ln L(\theta|y) = \sum_{i=1}^{T}\left[\ln\left(\left(\frac{n!}{y_i!\,(n-y_i)!}\right)\theta^{y_i}(1-\theta)^{n-y_i}\right)\right]$$

$$\ln L(\theta|y) = \sum_{i=1}^{T}\left[\ln\left(\frac{n!}{y_i!\,(n-y_i)!}\right) + \ln(\theta^{y_i}) + \ln(1-\theta)^{n-y_i}\right]$$

$$\ln L(\theta|y) = \sum_{i=1}^{T}\left[\ln\left(\frac{n!}{y_i!\,(n-y_i)!}\right)\right] + \sum_{i=1}^{T}[\ln(\theta^{y_i})] + \sum_{i=1}^{T}[\ln(1-\theta)^{n-y_i}]$$

$$\ln L(\theta|y) = \sum_{i=1}^{T}\left[\ln\left(\frac{n!}{y_i!\,(n-y_i)!}\right)\right] + \ln(\theta)\sum_{i=1}^{T}[y_i] + \ln(1-\theta)\sum_{i=1}^{T}[n-y_i]$$

4. Maximize the expression obtained in 3 and derive the estimated parameter of interest.

**Note:**

- $\frac{\partial}{\partial x}\ln(u) = \frac{1}{u}\frac{\partial u}{\partial x}$

$$\frac{d\ln L(\theta|y)}{d\theta} = \frac{\sum(y_i)}{\theta} - \frac{\sum(n-y_i)}{1-\theta}$$

**FOC:**

$$\frac{d\ln L(\theta|y)}{d\theta} = \frac{\sum(y_i)}{\tilde{\theta}} - \frac{\sum(n-y_i)}{1-\tilde{\theta}} = 0$$

$$\frac{\sum(y_i)}{\tilde{\theta}} = \frac{\sum(n-y_i)}{1-\tilde{\theta}}$$

$$\frac{1-\tilde{\theta}}{\tilde{\theta}} = \frac{\sum(n-y_i)}{\sum(y_i)}$$

$$\frac{1}{\tilde{\theta}} - 1 = \frac{nT - \sum(y_i)}{\sum(y_i)}$$

$$\frac{1}{\tilde{\theta}} = \frac{nT}{\sum(y_i)}$$

$$\tilde{\theta} = \frac{\sum(y_i)}{n*T} = \bar{Y}\,; \quad T\ Observations, n\ trials$$

**4. Uniform $\boldsymbol{U(0,b)}$**

1. We know our density function is:

$$f(y|b) = \begin{cases} \frac{1}{b} \; if \; 0 \leq y \leq b \\ 0 \; otherwise \end{cases}$$

Or

$$f(y) = \frac{1}{b} * I(0 \leq y \leq b)$$

   - $I(0 \leq y \leq b) = \begin{cases} 1 \; if \; 0 \leq y \leq b \\ 0 \; if \; \mathrm{y} > b \; or \; y < 0 \end{cases}$

2. We obtain the Joint Density Function:

$$f(y_1, y_2, \dots, y_T|b) = L(b|y_1, y_2, \dots, y_T) = L(b|\boldsymbol{y}) = \prod_{i=1}^{T} \left[\frac{1}{b}\right] * I(\max(y_1, \dots, y_T) \leq b)$$

$$L(b|\boldsymbol{y}) = \frac{1}{b^T} * I(\max(y_1, \dots, y_T) \leq b)$$

   - $I(\max(y_1, \dots, y_T) \leq b) = \mathrm{I}(y_1, \dots, y_T \in [0, b]) = \begin{cases} 1 \; if \;\; \max(y_1, \dots, y_T) \leq b \\ 0 \; if \;\; \max(y_1, \dots, y_T) > b \end{cases}$

3. Maximize the likelihood

$I(\max(y_1, \dots, y_T) \leq b)$ places a restriction on the likelihood function. Without it, the likelihood would be maximized with $\hat{b} = 0$. However, the restriction allows us only to take the greater value of between the observed $y_i$'s (this is such in order to make the interval from 0 to b long enough to contain all data observed. One can notice that the MLE will be the maximum of these observations, that is:

$$\hat{b} = \max(y_1, \dots, y_T)$$

**5. Geometric (the probability distribution of the number X of Bernoulli trials needed to get one success, supported on the set { 1, 2, 3, ...})**

1. We know our PMF is:

$$f(y|\theta) = (1-p)^{y-1}p$$

2. We obtain the Joint Density Function:

$$f(y_1, y_2, \dots, y_T|\theta) = L(\theta|y_1, y_2, \dots, y_T) = L(\theta|\mathbf{y}) = \prod_{i=1}^{T}[(1-p)^{y_i-1}p]$$

3. Take the natural logarithm of the Joint Density Function (monotonic transformation):

$$\ln L(\theta|y) = \sum_{i=1}^{T}[\ln((1-p)^{y_i-1}p)]$$

$$\ln L(\theta|y) = \sum_{i=1}^{T}[\ln(p) + \ln((1-p)^{y_i-1})]$$

$$\ln L(\theta|y) = \sum_{i=1}^{T}[\ln(p)] + \sum_{i=1}^{T}[\ln((1-p)^{y_i-1})]$$

$$\ln L(\theta|y) = \ln(p) * T + \ln(1-p)\sum_{i=1}^{T}[y_i - 1]$$

4. Maximize the expression obtained in 3 and derive the estimated parameter of interest.

**Note:**

- $\frac{\partial}{\partial x}\ \ln(u) = \frac{1}{u}\frac{\partial u}{\partial x}$

$$\frac{d \ln L(\theta|y)}{d\theta} = \frac{T}{p} - \frac{\sum(y_i - 1)}{1-p}$$

**FOC:**

$$\frac{d \ln L(\theta|y)}{d\theta} = \frac{T}{\tilde{p}} - \frac{\sum(y_i - 1)}{1-\tilde{p}} = 0$$

$$\frac{1-\tilde{p}}{\tilde{p}} = \frac{\sum(y_i - 1)}{T}$$

$$\frac{1}{\tilde{p}} - 1 = \frac{\sum(y_i) - T}{T}$$

$$\frac{1}{\tilde{p}} = \frac{\sum(y_i)}{T}$$

$$\tilde{p} = \frac{T}{\sum(y_i)} = \frac{1}{\bar{y}}$$